The su(1,1) Tavis-Cummings model

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Abstract. A generic su(1,1) Tavis–Cummings model is solved both by the quantum inverse method and within a conventional quantum-mechanical approach. Examples of corresponding quantum dynamics including squeezing properties of the su(1,1) Perelomov coherent states for the multiatom case are given.

1. Introduction

Since Dicke published his work, in 1954, on coherence in spontaneous radiation process [1], the kind of models he introduced have received much attention, both theoretical and experimental.

In quantum optics the model which describes one two-level atom interacting with a single mode of a cavity field, solved exactly by Jaynes and Cummings [2], has become very popular especially after the introduction of the micromaser [3]. A generalization of the Jaynes–Cummings (JC) model to the case of $\mathcal{N}_0$ atoms was later solved exactly by Tavis and Cummings [4] at exact resonance, and by Hepp and Lieb for finite detuning [5].

The single-atom JC model has also been generalized to include a nonideal cavity (Kerr nonlinearity [6]), a dynamic Stark shift [6, 7], two-photon transitions [8] and an intensity-dependent coupling of the atom and the field [9–11]. Very recently the Tavis–Cummings (TC) model has also been generalized to include either a nonideal cavity or a dynamical Stark shift. This generalization was solved exactly by using the quantum inverse method [12]. Admittedly the rotating wave approximation, which was used in the derivation of JC and TC models, puts certain restrictions on the scope of the applicability of these models. For more general treatments, which go beyond the rotating wave approximation, see for example [13–15].

In this paper we report the exact solution of another generalization of the many-atom TC model. Now the usual boson operators, which describe the single mode of a cavity field, are replaced by operators $K_\pm$, K_0 satisfying the su(1,1) algebra

$$[K_0, K_\pm] = \pm K_\pm \qquad [K_-, K_+] = 2K_0. \tag{1}$$

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A motivation for this generalization is that there are different realizations of the su(1,1) algebra in terms of the usual boson operators, and by solving the more general model one can simultaneously solve a number of models with different field-atom interactions. In this paper we shall concentrate on two particular realizations, given in equations (8) and (9) below, which respectively describe the case of intensity-dependent coupling and the case of two-photon transitions.

We are thus led to study the Hamiltonian

$$H = \omega(K_0 - \kappa) + \omega_0 S_3 + g(K_+ S_- + K_- S_+) \tag{2}$$

where ω is the frequency of the cavity mode, ω_0 is the frequency of the atomic transition, g is the coupling strength between the atoms and the cavity mode, and κ is the so-called Bargmann index which classifies the realizations of the su(1,1) algebra equation (1). Operators K_0, $K_\pm$ are generators of the su(1,1) algebra.

The operators S_3, $S_\pm$ are collective spin variables of $\mathcal{N}_0$ two-level atoms,

$$S_\pm = \sum_{i=1}^{\mathcal{N}_0} \sigma_i^\pm \qquad S_3 = \frac{1}{2} \sum_{i=1}^{\mathcal{N}_0} \sigma_i^3$$

which satisfy the usual su(2) algebra

$$[S_3, S_\pm] = \pm S_\pm \qquad [S_+, S_-] = 2S_3.$$

Up to minor details the structure of the quantum space of the model coincides with that of the conventional TC model and is well described in [4]. Namely, the quantum space of the system of $\mathcal{N}_0$ two–level atoms $\mathcal{H}_{\mathcal{N}_0}$ is given by

$$\mathcal{H}_{\mathcal{N}_0} = \bigoplus k_r \mathcal{H}_r \qquad r = \frac{\mathcal{N}_0}{2}, \frac{\mathcal{N}_0}{2} - 1, \ldots, \varepsilon_{\mathcal{N}_0} \tag{3}$$

where $\varepsilon_{\mathcal{N}_0} = \frac{1}{4}(1 - (-1)^{\mathcal{N}_0})$. Here r is Dicke's occupation number, and $\mathcal{H}_r \simeq \mathcal{C}^{2r+1}$ is a complex space of dimension $(2r + 1)$ corresponding to an irreducible representation of the su(2) algebra given by

$$|m, r\rangle = \left(\frac{(r - m)!}{(r + m)!(2r)!} \right)^{\frac{1}{2}} (S_+)^{r+m} | - r, r\rangle$$
$$S_\pm |m, r\rangle = \sqrt{r(r + 1) - m(m \pm 1)} |m \pm 1, r\rangle$$
$$S_3 |m, r\rangle = m |m, r\rangle \qquad -r \leqslant m \leqslant r \tag{4}$$
$$S^2 |m, r\rangle = r(r + 1) |m, r\rangle$$
$$S^2 = (S_3)^2 + \tfrac{1}{2}(S_+ S_- + S_- S_+).$$

In equation (3) the number

$$k_r = \frac{\mathcal{N}_0!(2r + 1)}{(\frac{1}{2}\mathcal{N}_0 + r + 1)!(\frac{1}{2}\mathcal{N}_0 - r)!}$$

reflects the multiplicity of choices which can be used to create an atomic configuration with given quantum numbers r and m, the latter of which is an eigenvalue of S_3 while the former is the Dicke's occupation number. Thus the quantum space $\mathcal{H}$ of the model in equation (2) is given by $\mathcal{H} = \mathcal{H}_B \otimes \mathcal{H}_{\mathcal{N}_0}$, where $\mathcal{H}_B$ is the space of a representation of the su(1,1)

algebra characterized by the Bargmann index κ:

$$|n; \kappa\rangle = \left(\frac{\Gamma(2\kappa)}{n!\Gamma(2\kappa + n)}\right)^{\frac{1}{2}} K_+^n |0; \kappa\rangle$$

$$K^2 |n; \kappa\rangle = \kappa(\kappa - 1)|n; \kappa\rangle$$

$$K_0 |n; \kappa\rangle = (n + \kappa)|n; \kappa\rangle \tag{5}$$

$$K_+ |n; \kappa\rangle = \sqrt{(n + 1)(n + 2\kappa)}|n + 1; \kappa\rangle$$

$$K_- |n; \kappa\rangle = \sqrt{n(n + 2\kappa - 1)}|n - 1; \kappa\rangle.$$

Here

$$K_- |0, \kappa\rangle = 0 \tag{6}$$

and K^2 is the Casimir operator of the su(1,1) algebra,

$$K^2 = K_0^2 - \tfrac{1}{2}(K_+ K_- + K_- K_+). \tag{7}$$

In this work we shall concentrate on two particular realizations of the su(1,1) algebra, the first of which has $\kappa = \tfrac{1}{2}$ and

$$K_+ = i\sqrt{N}a^\dagger \qquad K_- = -ia\sqrt{N} \qquad K_0 = N + \tfrac{1}{2}. \tag{8}$$

This case corresponds to an intensity-dependent TC model. This model was introduced for the first time for the one-atom case by Buck and Sukumar [10]. It is obvious that the coupling constant between the atoms and the monochromatic field in the Hamiltonian equation (2) is in a way proportional to $\sqrt{N}$, should the realization of the su(1,1) algebra given by equation (8) be applied. This is what provides a motivation to characterize this model as *intensity dependent*. Even though such a model may seem rather artificial, it is of theoretical interest since its quantum dynamics always shows in the one-atom case strictly periodic behaviour such that all the physical quantities involved are expressed in closed algebraic form via trigonometric functions. We show that this property is already violated in the case of two atoms (for details see section 3) while a kind of quasiperiodic behaviour can still be observed.

The second realization we shall consider here,

$$K_+ = \tfrac{1}{2}(a^\dagger)^2 \qquad K_- = \tfrac{1}{2}a^2 \qquad K_0 = \tfrac{1}{2}a^\dagger a + \tfrac{1}{4} \tag{9}$$

corresponds to $\kappa = \tfrac{1}{4}$ and $\kappa = \tfrac{3}{4}$ which means that the representation space $\mathcal{H}_B$ in this case is decomposed into the direct sum

$$\mathcal{H}_B = \mathcal{H}_B^{1/4} \oplus \mathcal{H}_B^{3/4}$$

of the two irreducible components $\mathcal{H}_B^{1/4}$ and $\mathcal{H}_B^{3/4}$, where the former is spanned by the states with an even number of quanta and the latter by the states with an odd number of quanta.

This case corresponds to a TC model in which atomic transitions are mediated by two photons: $a^\dagger$, a are the usual operators of the Heisenberg–Weyl algebra: $[a, a^\dagger] = 1$, $a^\dagger a = N$. Such a model in the one-atom case is very widely used in the theory, for example of the two-photon micromaser [16–18] in which a quantized field in a high Q-cavity is successively pumped by a stream of Rydberg atoms arranged such that only one atom at most is present in the cavity at any instant of time. The Hamiltonian describing this model is

$$H_{tp} = \left(\omega_0 + \frac{\xi}{2}a^\dagger a\right)\sigma^3 + \frac{\omega}{2}a^\dagger a + \frac{g}{2}(\sigma^+ a^2 + \sigma^-(a^\dagger)^2) \tag{10}$$

where the term $\frac{\xi}{2}a^\dagger a\sigma^3$ describes a dynamical Stark shift. Our many-atoms model provides an obvious generalization of this experimental situation to the case when $\mathcal{N}_0$ Rydberg atoms are present in the cavity at any given instant of time. In the following section we show, among other things, that an extension of our many-atoms model to include the dynamical Stark shift term is quite straightforward.

This paper is organized as follows. In section 2 we describe the solution of the model by the quantum inverse method (QIM) and compare the results against those of a more conventional quantum-mechanical approach along the lines similar to [4].

In section 3, we will apply the results of the exact solution to be derived in section 2. We study the interaction of the squeezed light described as the su(1,1) Perelomov coherent states with $\mathcal{N}_0$ two-level atoms for the simplest case $\mathcal{N}_0 = 2$. We will confine our considerations to the case most interesting for applications, which is that of atomic transitions mediated by two photons, and will show that quasiperiodic decays and revivals of squeezing can still be observed in the case of two atoms. We will also investigate the time evolution of atomic inversion in the case when the field is initially in the su(1,1) Perelomov coherent state, while the atomic subsystem is in its ground state. The general formulae will be derived together with a detailed evaluation of the case $\mathcal{N}_0 = 2$. We will show that the generalization to the case of two atoms of the JC model [9–11] with intensity-dependent coupling does not completely destroy the periodic behaviour of the inversion.

2. Exact solution by the quantum inverse method

In this section we consider the eigenvalue problem

$$H|\nu\rangle = E^\nu|\nu\rangle \tag{11}$$

for the Hamiltonian equation (2) and show how it can be solved exactly. In order to obtain the complete set of eigenfunctions $|\nu\rangle$ (ν is a set of quantum numbers) and the corresponding eigenvalues E^ν, we shall first employ here a powerful machinery of the QIM [19]. Up to minor modifications required by the su(1,1) algebra, the classification of the eigenstates by a set of quantum numbers ν can be found in [4]. This suggests that we look for eigenfunctions of the Hamiltonian equation (2) as a superposition of simultaneous eigenfunctions of the number operator

$$\hat{M}_r = S_3 + K_0 - \kappa + r$$
$$\hat{M}_r|n; \kappa\rangle \otimes |m, r\rangle = (n + m + r)|n; \kappa\rangle \otimes |m, r\rangle \tag{12}$$

and those of the unperturbed Hamiltonian

$$H_0 = \omega(K_0 - \kappa) + \omega_0 S_3 \tag{13}$$
$$H_0|n; \kappa\rangle \otimes |m, r\rangle = (\omega n + \omega_0 m)|n; \kappa\rangle \otimes |m, r\rangle \tag{14}$$

where

$$n + m + r = M \qquad M \geqslant 0. \tag{15}$$

More specifically, we shall look for eigenstates in the form

$$|\nu\rangle = \sum_{n=n_0}^{M} A_{n+r}^{M,r,j}|n, \kappa\rangle \otimes |M - n - r, r\rangle \tag{16}$$

for each block $\mathcal{H}_r$ in equation (3). Here the quantum number j is

$$j = M - n_0 + 1 \tag{17}$$

which is the number of states for the given M and r. Here

$$n_0 = \max(M - 2r, 0).$$

Thus the set of quantum numbers characterizing the basis of each space $\mathcal{H}_r$ for the given eigenvalues of $\hat{M}_r$, M is $\nu = (M, r, j)$. The necessary quantum numbers now defined, we next explain how to consider the problem specified by equation (11) in the framework of QIM.

To begin with let us consider [19] a bilinear intertwining relation

$$R(\lambda - \mu)L(\lambda) \otimes L(\mu) = L(\mu) \otimes L(\lambda)R(\lambda - \mu) \tag{18}$$

where $\lambda, \mu \in \mathcal{C}$, with the rational [19] R-matrix

$$R(\lambda - \mu) = \begin{pmatrix} f(\mu, \lambda) & 0 & 0 & 0 \\ 0 & g(\mu, \lambda) & 1 & 0 \\ 0 & 1 & g(\mu, \lambda) & 0 \\ 0 & 0 & 0 & f(\mu, \lambda) \end{pmatrix} \tag{19}$$

$$f(\mu, \lambda) = \frac{\mathrm{i}(\lambda - \mu) + 1}{\mathrm{i}(\lambda - \mu)} \qquad g(\mu, \lambda) = \frac{1}{\mathrm{i}(\lambda - \mu)}. \tag{20}$$

Here $L(\lambda)$ is a quantum operator of matrix dimension 2,

$$L(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \tag{21}$$

Its entries are quantum operators acting on the quantum space $\mathcal{H}_B \otimes \mathcal{H}_r$. This space has the vacuum vector

$$|0\rangle_r = |0; \kappa\rangle \otimes |-r, r\rangle. \tag{22}$$

We now apply QIM in its algebraic Bethe ansatz form [19]. For convenience in the appendix we provide a brief account of the method to the extent necessary for this work. It is worth mentioning however that in practice we use the commutation relations equation (18). It can be readily seen (cf the appendix) that the eigenfunctions of the trace

$$\tau(\lambda) = \mathrm{tr}L(\lambda) = A(\lambda) + D(\lambda) \tag{23}$$

in the quantum space $\mathcal{H}_B \otimes \mathcal{H}_r$ are constructed through expressions (up to an arbitrary constant $\phi_0^{M\alpha}$)

$$|M, \alpha; \{\lambda_i^\alpha\}_{i=1}^M\rangle = \phi_0^{M,\alpha} \prod_{i=1}^M B(\lambda_i^\alpha)|0\rangle_r \tag{24}$$

while the corresponding eigenenergies are given by

$$\tau(\lambda)|M, \alpha; \{\lambda_i^\alpha\}_{i=1}^M\rangle = \theta^{M,\alpha}(\lambda)|M, \alpha; \{\lambda_i^\alpha\}_{i=1}^M\rangle \tag{25}$$

with

$$\theta^{M,\alpha}(\lambda) = \theta(\lambda, \{\lambda_i^\alpha\}_{i=1}^M) = a(\lambda) \prod_{i=1}^M f(\lambda, \lambda_i^\alpha) + d(\lambda) \prod_{i=1}^M f(\lambda_i^\alpha, \lambda). \tag{26}$$

Provided that the set of numbers $\{\lambda_i^\alpha\}_{i=1}^M$ satisfy the system of Bethe equations

$$\frac{d(\lambda_n^\alpha)}{a(\lambda_n^\alpha)} = \prod_{\substack{i \neq n \\ i=1}}^M \frac{\mathrm{i}(\lambda_i^\alpha - \lambda_n^\alpha) + 1}{\mathrm{i}(\lambda_i^\alpha - \lambda_n^\alpha) - 1} \tag{27}$$

where the superscript α denotes a set of auxiliary quantum numbers that will be specified below equation (35), $a(\lambda)$ and $d(\lambda)$ are eigenvalues of the operators $A(\lambda)$ and $D(\lambda)$ defined in equation (21), and the respective eigenfunction is the vacuum $|0\rangle_r$:

$$A(\lambda)|0\rangle_r = a(\lambda)|0\rangle_r \qquad D(\lambda)|0\rangle_r = d(\lambda)|0\rangle_r. \tag{28}$$

Notice that equation (26) is in fact equivalent to a linear finite-difference spectral problem (Baxter equation) [20, 21],

$$\theta^{M,\alpha}(\lambda)Q_M(\lambda) = a(\lambda)Q_M(\lambda + \mathrm{i}) + d(\lambda)Q_M(\lambda - \mathrm{i}) \tag{29}$$

for a polynomial $Q_M(\lambda)$ determined by its zeros $\{\lambda_i^\alpha\}_{i=1}^M$:

$$Q_M(\lambda) = \prod_{i=1}^{M}(\lambda - \lambda_i^\alpha).$$

If we now choose a solution of the bilinear intertwining relation equation (18) with the R-matrix (19) and with the L-operator [19]

$$l(\lambda) = l_s(\lambda)l_k(\lambda)$$
$$= \begin{pmatrix} \mathrm{i}\lambda - \epsilon - S_3 & -S_+ \\ S_- & -(\mathrm{i}\lambda - \epsilon + S_3) \end{pmatrix} \begin{pmatrix} \mathrm{i}\lambda - K_0 & K_+ \\ -K_- & \mathrm{i}\lambda + K_0 \end{pmatrix} \tag{30}$$

it is evident (cf equations (2), (23), (30)) that the Hamiltonian equation (2) can be expressed in the form

$$H = g\tau(\lambda_0) - \kappa\omega \tag{31}$$

where

$$\lambda_0 = \frac{\mathrm{i}\omega_0}{2g} \qquad \epsilon = \frac{\omega - \omega_0}{2g}. \tag{32}$$

This means that the eigenfunctions of the Hamiltonian equation (2) are constructed through equation (24) with the operator

$$B(\lambda) = (\mathrm{i}\lambda - \epsilon - S_3)K_+ - S_+(\mathrm{i}\lambda + K_0) \tag{33}$$

while the corresponding eigenvalues E^ν are given by

$$E^\nu = g\theta^{M,\alpha}(\lambda_0) - \kappa\omega \tag{34}$$

with

$$a(\lambda) = (\mathrm{i}\lambda - \epsilon + r)(\mathrm{i}\lambda - \kappa) \qquad d(\lambda) = -(\mathrm{i}\lambda - \epsilon - r)(\mathrm{i}\lambda + \kappa). \tag{35}$$

Hereafter the set of quantum numbers α is specified as $\alpha = (r, j)$. To summarize, the algorithm for solving the eigenvalue problem equation (11) via QIM is as follows. Solve the Bethe equations (27) with (35) for given M and $\alpha = (r, j)$, construct the system of eigenfunctions through equation (24) with (33), and find the eigenenergies through equation (34).

To exemplify this algorithm we have solved the Bethe equations for $M = 3$ by the Newton–Raphson method. The results are given in table 1.

We explain next how our generic model can be extended to the case when the dynamical Stark shift is taken into account.

It is obvious that any c-number matrix K satisfies the bilinear intertwining relation equation (18) with the rational R-matrix equation (19), namely,

$$R(\lambda - \mu)K \otimes K = K \otimes K R(\lambda - \mu).$$

Table 1. Solutions to the Bethe equations and the corresponding eigenenergies for $\omega = 1.0$, $\epsilon = 0.2$, $g = 1.0$ and $\kappa = 0.5$.

r	0	$\frac{1}{2}$	1	$\frac{3}{2}$
j	1	2	3	4
$\lambda_1^{r,1}$	$-1.118\,03$	$-0.632\,329 - 0.487\,085i$	$0.823\,284i$	$-0.231\,329 + 1.316\,88i$
$\lambda_2^{r,1}$	0	$0.632\,329 - 0.487\,085i$	$0.071\,677\,5i$	$0.231\,329 + 1.316\,88i$
$\lambda_3^{r,1}$	$1.118\,03$	$-0.629\,160i$	$-1.171\,48i$	$0.509\,875i$
$E^{3,r,1}$	3.0	$-0.506\,66$	$1.846\,96$	$8.387\,21$
$\lambda_1^{r,2}$		$-0.771\,591 + 0.492\,254i$	$-0.559\,214i$	$-0.489\,366i$
$\lambda_2^{r,2}$		$0.771\,591 + 0.492\,254i$	$-0.223\,311 - 1.079\,50i$	$-1.259\,67i$
$\lambda_3^{r,2}$		$0.418\,821i$	$+0.223\,311 - 1.079\,50i$	$-1.768\,40i$
$E^{3,r,2}$		$5.506\,66$	$-3.036\,45$	$-4.934\,87$
$\lambda_1^{r,3}$			$-0.493\,470 + 0.925\,053i$	$1.153\,95i$
$\lambda_2^{r,3}$			$+0.493\,470 + 0.925\,053i$	$-0.590\,669i$
$\lambda_3^{r,3}$			$0.544\,639i$	$-1.670\,62i$
$E^{3,r,3}$			$7.189\,48$	$-0.114\,678$
$\lambda_1^{r,4}$				$1.335\,12i$
$\lambda_2^{r,4}$				$0.436\,292i$
$\lambda_3^{r,4}$				$-1.490\,26i$
$E^{3,r,4}$				$2.662\,32$

This allows us to modify the L-operator equation (30) such that

$$\tilde{L}(\lambda) = K L_S(\lambda) K L_K(\lambda) \tag{36}$$

with $K = \exp\left(\frac{\gamma}{2}\sigma^3\right)$, $\gamma \geqslant 0$. The trace equation (23) is then modified, and becomes

$$\tilde{\tau}(\lambda) = -2\sinh(\gamma)\lambda^2 + J_1\lambda + J_0 \tag{37}$$

where

$$J_0 = 2\tilde{\epsilon}\cosh(\gamma)K_0 + 2\sinh(\gamma)S_3K_0 + S_+K_- + S_-K_+$$
$$J_1 = -2i\tilde{\epsilon}\sinh(\gamma) - 2i\cosh(\gamma)(S_3 + K_0). \tag{38}$$

Since $[\tau(\lambda), \tau(\mu)] = 0$ for $\forall \lambda, \mu \in \mathcal{C}$ [19], the operators J_0 and J_1 are integrals of motion and commute. As can be readily seen their linear combination

$$\tilde{H} = \alpha_1 J_1 + \alpha_0 J_0 + c = \alpha_1 \left.\frac{d\tilde{\tau}(\lambda)}{d\lambda}\right|_{\lambda=0} + \alpha_0 \tilde{\tau}(0) + c$$

is equal to the expression

$$\tilde{H} = \omega(K_0 - \kappa) + (\omega_0 + \xi(K_0 - \kappa))S_3 + g(K_+S_- + K_-S_+) \tag{39}$$

provided that the parameters γ and $\tilde{\epsilon}$ are chosen such that

$$\xi = 2g\sinh(\gamma) \qquad \tilde{\epsilon} = \frac{\omega - \omega_0}{2g\cosh(\gamma)} + \kappa\tanh(\gamma)$$

while

$$\alpha_1 = \frac{i\omega_0}{2\cosh(\gamma)} - i\kappa g \tanh(\gamma) \qquad \alpha_0 = g$$

$$c = -\omega\kappa - 2\frac{\tanh(\gamma)}{g\cosh(\gamma)}\left(\frac{1}{4}\omega_0(\omega - \omega_0) - \kappa^2 g^2 \sinh(\gamma)^2 + \kappa g\left(\omega_0 - \frac{\omega}{2}\right)\sinh(\gamma)\right).$$ (40)

It is obvious that when $\kappa = \frac{1}{4}, \frac{3}{4}$ the operator equation (39) is a many-atoms generalization of the Hamiltonian equation (10). The solution of the model equation (39) by QIM can be realized by replacing the vacuum eigenvalues $a(\lambda)$ and $d(\lambda)$ (cf equations (28) and (35)) in the expressions equations (26), (27) and (29) by their modified analogues, namely

$$\tilde{a}(\lambda) = e^\gamma (i\lambda - \tilde{\epsilon} + r)(i\lambda - \kappa) \qquad \tilde{d}(\lambda) = -e^{-\gamma}(i\lambda - \tilde{\epsilon} - r)(i\lambda + \kappa).$$ (41)

We shall not elaborate on the dynamical Stark effect any further here, and postpone investigation of its influence on quantum dynamics to a forthcoming publication.

In order to make connection between the eigenfunctions in the form of equation (16) and those in the form of equation (24), we construct the eigenfunctions of $\tau(\lambda)$ (equation (25)) belonging to the space $\mathcal{H}_B \otimes \mathcal{H}_r$ such that

$$|M, \alpha\rangle = \sum_{n=n_0}^{M} \mathcal{A}_{n+r}^{M,\alpha}|n; \kappa\rangle \otimes |M - n - r, r\rangle.$$ (42)

The recurrence relation

$$\mathcal{A}_{n+r+1}^{M,\alpha} = c_{n+r}\mathcal{A}_{n+r}^{M,\alpha} + f_{n+r-1}\mathcal{A}_{n+r-1}^{M,\alpha} \qquad n_0 \leqslant n \leqslant M - 1$$ (43)

along with the condition

$$\mathcal{A}_{M+r}^{M,\alpha} = \tilde{c}_{M+r-1}\mathcal{A}_{M+r-1}^{M,\alpha}$$ (44)

where

$$c_{n+r} = p_n/\sqrt{((n+1)(n+2\kappa)(M-n)(2r+1-M+n))}$$ (45)

$$f_{n+r-1} = -\sqrt{\frac{n(n+2\kappa-1)(M-n+1)(2r-M+n)}{(n+1)(n+2\kappa)(M-n)(2r+1-M+n)}}$$ (46)

$$\tilde{c}_{M+r-1} = \sqrt{2rM(M-1+2\kappa)}/p_M$$ (47)

$$p_n = -2\epsilon(n+\kappa) + p^{M,\alpha}$$ (48)

$$\theta^{M,\alpha}(\lambda) = -2i\lambda(M+\kappa-r) + p^{M,\alpha}$$ (49)

can now be easily obtained by substituting the eigenfunction equation (42) to the eigenvalue problem equation (25).

The recurrence relation equation (43) can be solved by chain fractions and we find that

$$\mathcal{A}_{n+r+1}^{M,\alpha} = \mathcal{A}_{n_0+r}^{M,\alpha}\prod_{k=n_0}^{n} c_{k+r}y_{k+r} \qquad n_0 \leqslant n \leqslant M - 1$$ (50)

where y_{k+r}, $n_0 \leqslant k \leqslant n$, are given by

$$y_{k+r} = 1 + \cfrac{d_{k+r-1}}{1 + \cfrac{d_{k+r-2}}{1 + \cfrac{\ddots}{1 + \cfrac{d_{n_0+r+1}}{1 + d_{n_0+r}}}}}$$ (51)

$$d_{n+r-1} = -n(n-1+2\kappa)(M-n+1)(2r-M+n)/p_n p_{n-1}.$$ (52)

Comparing equation (50) for $n = M - 1$ with equation (44), we arrive at an algebraic equation for the eigenvalue problem equation (25),

$$\frac{2rM(M - 1 + 2\kappa)}{p_M p_{M-1}} = y_{M+r-1}. \tag{53}$$

The eigenenergies of the eigenvalue problem equation (11) are then found from equation (34), while the amplitudes $A_{n+r}^{M,r,j}$ defined by equation (16) can be identified with those given by equation (50),

$$A_{n+r}^{M,r,j} = \mathcal{A}_{n+r}^{M,\alpha}. \tag{54}$$

Note that $[\tau(\lambda), H] = 0$.

Identifying L-operators $L_1(\lambda)$ and $L_2(\lambda)$ of (A.11) with $L_S(\lambda)$ and $L_K(\lambda)$ (cf equation (30)), respectively, we can further specify the quantities involved in the representation of the eigenfunctions as given by (A.13) such that

$$a_1(\lambda) = (i\lambda - \epsilon + r) \qquad d_2(\lambda) = i\lambda + \kappa \qquad B_1(\lambda) = -S_+ \qquad B_2(\lambda) = K^+ \tag{55}$$

which allows for a reformulation of the eigenfunctions equation (A.13) in the form

$$|M, \alpha; \{\lambda_i^\alpha\}_{i=1}^M\rangle = \phi_0^{M,\alpha} \sum_{n=n_0}^M Q_{n,M-n-r} \sum_{\{\lambda_i^\alpha\}_{i=1}^M = \{\lambda_{i_s}^\alpha\}_{s=1}^n \cup \{\lambda_{i_p}^\alpha\}_{p=n+1}^M} a_1(\lambda_{i_s}^\alpha) d_2(\lambda_{i_p}^\alpha) f(\lambda_{i_s}^\alpha, \lambda_{i_p}^\alpha)$$
$$|n, \kappa\rangle \otimes |M - n - r, r\rangle \tag{56}$$

where

$$Q_{n,M-n-r} = (-1)^{M-n} \left(\frac{\Gamma(2\kappa)}{n!\Gamma(2\kappa + n)} \frac{(2r - M + n)!}{(M - n)!(2r)!} \right)^{-\frac{1}{2}}. \tag{57}$$

Comparing equation (56) with equation (42), we arrive at the result

$$A_{n+r}^{M,\alpha} = \phi_0^{M,\alpha} Q_{n,M-n-r} \sum_{\{\lambda_i^\alpha\}_{i=1}^M = \{\lambda_{i_s}^\alpha\}_{s=1}^n \cup \{\lambda_{i_p}^\alpha\}_{p=n+1}^M} (i\lambda_{i_s}^\alpha - \epsilon + r)(i\lambda_{i_p}^\alpha + \kappa) \frac{i(\lambda_{i_p}^\alpha - \lambda_{i_s}^\alpha) + 1}{i(\lambda_{i_p}^\alpha - \lambda_{i_s}^\alpha)}. \tag{58}$$

The summation in equation (58) is explained in the appendix.

We shall exemplify the application of these results in section 3 below while discussing the corresponding quantum dynamics. First we shall discuss the limit of a great number of particles $\mathcal{N}_0$ and of a great occupation number r. To be more exact we consider the limit

$$\mathcal{N}_0, r \longrightarrow \infty \tag{59}$$

while M is kept finite, $n_0 = 0$ and $j = 1, 2, \ldots, M + 1$ (cf equation (17)).

The asymptotic behaviour of the roots $\{\lambda_i^{r,j}\}_{i=1}^M$ is in this limit

$$\lambda_i^{r,j} = \sqrt{r} \zeta_i^j + \mathcal{O}(1) \tag{60}$$

as can be deduced from the Bethe equations (27).

The corresponding limiting form of the Baxter equation (29) is

$$\frac{d^2 P(\zeta)}{d\zeta^2} + 2\left(\zeta + \frac{\kappa}{\zeta}\right) \frac{dP(\zeta)}{d\zeta} - \left(2M + \frac{q^j}{\zeta}\right) P(\zeta) = 0 \tag{61}$$

where the polynomial $P(\zeta)$ is defined by its zeros $\{\zeta_i^j\}_{i=1}^M$:

$$P(\zeta) = \prod_{i=1}^M (\zeta - \zeta_i^j). \tag{62}$$

The parameters q_j, $j = 1, 2, \ldots, M + 1$, in equation (62) are defined by equation (53) in the limit (59), i.e. as roots of the polynomial equation

$$0 = 1 + \frac{\alpha_M}{q_j^2} + \cfrac{\dfrac{\alpha_{M-1}}{q_j^2}}{1 + \cfrac{\dfrac{\alpha_{M-2}}{q_j^2}}{1 + \cfrac{\ddots}{1 + \cfrac{\dfrac{\alpha_2}{q_j^2}}{1 + \dfrac{\alpha_1}{q_j^2}}}}} \tag{63}$$

$$\alpha_n = 2n(n - 1 + 2\kappa)(M - n + 1) \qquad n = 1, 2, \ldots, M. \tag{64}$$

It is evident from equation (61) that the zeros $\{\zeta_i^j\}_{i=1}^M$ of the polynomial equation (62) satisfy

$$\zeta_n^j + \frac{\kappa}{\zeta_n^j} + \sum_{\substack{i \neq n \\ i=1}}^M \frac{1}{\zeta_n^j - \zeta_i^j} = 0 \qquad j = 1, 2, \ldots, M + 1. \tag{65}$$

This result can also be obtained from the Bethe equations (27) in the limit (59).

Connection between the eigenvalues of the eigenvalue problem equation (25) and the parameters q_j can be established through an asymptotic expansion, and we find in the limits equations (59) and (60) that

$$\frac{1}{\sqrt{r}} \theta^{M,r,j}(\sqrt{r}\zeta) = 2\mathrm{i}\zeta r - 2\mathrm{i}\zeta(M + \kappa) - \mathrm{i}q_j + \mathcal{O}\left(\frac{1}{\sqrt{r}}\right). \tag{66}$$

It is worth mentioning here that QIM, as applied to a JC-type model with $su(2) \oplus su(1, 1)$ dynamical algebra, was discussed in [22] in the limit leading to a Gaudin type of model [23, 24]. This limit leads to Bethe equations similar to our equation (65). The main difference between our approach and that of [22, 23] lies in our using the bilinear intertwining relation equation (18) instead of the so-called classical Yang–Baxter equation [19]

$$[l(\zeta) \otimes I, I \otimes l(z)] = [l(\zeta) \otimes I + I \otimes l(z), r(\zeta - z)] \tag{67}$$

where $r(\zeta - z)$ is the classical r-matrix [19], while $l(\zeta)$ is again a quantum operator of matrix dimension two. In order to make contact between these two approaches we replace $\lambda = \sqrt{r}\zeta$ and $\mu = \sqrt{r}z$ such that the R-matrix equation (19) takes the form

$$R(\zeta, z) = \Pi - \frac{\mathrm{i}}{\sqrt{r}(\zeta - z)} I_4 \tag{68}$$

where Π is the permutation matrix in $\mathcal{C}^2 \otimes \mathcal{C}^2$, $\Pi^2 = I_4$. Then in the limit equation (59),

$$\frac{S_+}{\sqrt{2r}} \to b^\dagger \qquad \frac{S_-}{\sqrt{2r}} \to b \qquad \frac{S_3}{r} \to -1 \tag{69}$$

where $b^\dagger$, b are usual boson operators (cf [25]). The L-operator equation (30) in this same limit has the expansion

$$-\frac{\mathrm{i}}{\zeta r^{\frac{3}{2}}} L(\zeta) = I - \frac{\mathrm{i}}{\sqrt{r}} l(\zeta) + \mathcal{O}\left(\frac{1}{r}\right) \tag{70}$$

with

$$l(\zeta) = \begin{pmatrix} X_3(\zeta) & X_+(\zeta) \\ X_-(\zeta) & -X_3(\zeta) \end{pmatrix} \tag{71}$$

$$X_3(\zeta) = -\zeta - \frac{K_0}{\zeta} \qquad X_+(\zeta) = \frac{K_+}{\zeta} - i\sqrt{2}b^\dagger \qquad X_-(\zeta) = -\frac{K_-}{\zeta} + i\sqrt{2}b. \tag{72}$$

Substituting equations (68) and (70) into equation (18) we arrive at equation (67) with the *r*-matrix

$$r(\zeta, z) = \frac{\Pi}{\zeta - z}. \tag{73}$$

The limiting form of the Hamiltonian equation (2) is

$$H_\infty = K_+ b + K_- b^\dagger$$

which for the realization of the su(1, 1) algebra given by equation (9) can be interpreted as a *trilinear oscillator* model.

3. Quantum dynamics

In this section we consider the time evolution of the su(1,1) coherent states, and compare their squeezing properties in the multiatomic case ($\mathcal{N}_0 > 1$) with those in the one-atom ($\mathcal{N}_0=1$) case discussed in [8]. In [8] Gerry studied the interaction of su(1,1)-squeezed light with a single two-level atom and reported periodic revivals of the initial squeezing at longer times. Here we apply the general solution obtained in the previous section to examine whether the periodic revivals of squeezing are still observed in the case of two atoms. We shall assume that the active atoms and the monochromatic field are initially prepared in a quantum state $|\Phi_0\rangle$,

$$|\Phi_0\rangle = \sum_{r=\varepsilon_{\mathcal{N}_0}}^{\frac{\mathcal{N}_0}{2}} \sum_{i=1}^{k_r} |\xi; \kappa\rangle \otimes | -r, r\rangle_i \tag{74}$$

where $|\xi; \kappa\rangle$ is the Perelomov su(1,1) coherent state [26, 27]

$$|\xi; \kappa\rangle = (1 - |\xi|^2)^\kappa \sum_{m=0}^{\infty} \left(\frac{\Gamma(m + 2\kappa)}{m!\Gamma(2\kappa)} \right)^{\frac{1}{2}} \xi^m |m; \kappa\rangle \tag{75}$$

with $|\xi| \leqslant 1$. In the initial state equation (74) the spin system is in the ground state, which is a sum (up to the degeneration number k_r) of the lowest vectors $|r, -r\rangle$ of each of the subspaces $\mathcal{H}_r$ defined in equation (3).

The time evolution of the initial state equation (74) can then be obtained by application of an evolution operator $U(t)$ that can be constructed through the complete set of eigenfunctions $|\nu\rangle$, $\nu = (M, r, j)$:

$$U(t) = \sum_\nu |\nu\rangle\langle\nu| \exp(-iE^\nu t). \tag{76}$$

The evolution of atomic inversion is then given by

$$\langle\Phi(t)|S_3|\Phi(t)\rangle = \sum_{\nu,\nu'} \langle\Phi_0|\nu'\rangle\langle\nu|\Phi_0\rangle \exp(i(E^{\nu'} - E^\nu)t)\langle\nu'|S_3|\nu\rangle \tag{77}$$

where $|\Phi(t)\rangle = U(t)|\Phi_0\rangle$.

If we employ in equation (77) the expression equation (16) for the eigenfunctions we find that

$$\langle \Phi(t)|S_3|\Phi(t)\rangle = \sum_{\substack{M,r \\ j,j'}} \bar{A}_{M+r}^{M,r,j} A_{M+r}^{M,r,j'} |\langle M; \kappa|\xi; \kappa\rangle|^2 \exp(\mathrm{i}(E^{M,r,j'} - E^{M,r,j})t)$$

$$\times \sum_{n=n_0}^{M} \bar{A}_{n+r}^{M,r,j'} A_{n+r}^{M,r,j} (M - r - n). \tag{78}$$

We shall exemplify the application of this formula in the simplest nontrivial case, which is that of two atoms, $\mathcal{N}_0 = 2$, $r = 0, 1$.

For simplicity we shall only consider the case of exact resonance between the field and the two–level atom, which means $\omega_0 = \omega$, i.e. $\epsilon = 0$.

In this case the evaluation of eigenfunctions $|\nu\rangle$ in the form of equation (16) is rather straightforward.

If $r = 0$, then $|M, 0, j\rangle = |M; \kappa\rangle \otimes |0, 0\rangle$, and $j = 1$.

If $r = 1$ and $M = 0$, then $|0, 1, j\rangle = |0; \kappa\rangle \otimes |1, -1\rangle$, and $j = 1$.

If $r = 1$ and $M = 1$, then $|1, 1, j\rangle = A_2^{1,1,j}|1; \kappa\rangle \otimes |-1, 1\rangle + A_1^{1,1,j}|0; \kappa\rangle \otimes |0, 1\rangle$, and $j = 1, 2$.

Solving in this particular case equations (44), (50) and (53) with the normalization condition $\langle 1, 1, j|1, 1, j\rangle = 1$, we find that

$$A_1^{1,1,j} = (-1)^j \frac{\sqrt{2}}{2} \qquad A_2^{1,1,j} = -\frac{\sqrt{2}}{2}$$
$$E^{1,1,j} = (-1)^{j+1} 2g\sqrt{\kappa}. \tag{79}$$

In the case $r = 1$, $M > 1$, $k = 1, 2$; $j = k, 3$; the eigenfunction is

$$|M > 1, 1, j\rangle = A_{M+1}^{M,1,j}|M; \kappa\rangle \otimes |-1, 1\rangle + A_M^{M,1,j}|M - 1; \kappa\rangle \otimes |0, 1\rangle$$
$$+ A_{M-1}^{M,1,j}|M - 2; \kappa\rangle \otimes |1, 1\rangle. \tag{80}$$

Note that $\langle M > 1, 1, j|M > 1, 1, j\rangle = 1$. In this case the solution of equations (44), (50) leads to the result

$$A_{M-1}^{M,1,k} = \sqrt{\frac{a_{M-1}}{2(a_M + a_{M-1})}} \qquad A_M^{M,1,k} = (-1)^{k+1}\frac{\sqrt{2}}{2} \qquad A_{M+1}^{M,1,k} = \sqrt{\frac{a_M}{2(a_M + a_{M-1})}} \tag{81}$$

where $a_M = M(2\kappa + M - 1)$ and

$$A_{M-1}^{M,1,3} = \sqrt{2} A_{M+1}^{M,1,k} \qquad A_M^{M,1,3} = 0 \qquad A_{M+1}^{M,1,3} = -\sqrt{2} A_{M-1}^{M,1,k}. \tag{82}$$

The equation for the eigenenergies equation (53) in this case of $M > 1$, $r = 1$, assumes the form

$$\frac{a_M}{\omega(M-1) - E^{M,1,j}} = \frac{1}{2g^2}(\omega(M-1) - E^{M,1,j}) - \frac{a_{M-1}}{\omega(M-1) - E^{M,1,j}} \tag{83}$$

which has the obvious solution

$$E^{M,1,k} = \omega(M-1) + (-1)^k g\sqrt{2(a_M + a_{M-1})} \qquad k = 1, 2$$
$$E^{M,1,3} = \omega(M-1). \tag{84}$$

When the results of equations (79), (81), (82) and (84) are taken into account in the general formula equation (78) for $\mathcal{N}_0 = 2$, this can be expressed in the form

$$\langle\Phi(t)|S_3|\Phi(t)\rangle = -|\langle\kappa; 0|\xi; \kappa\rangle|^2 - \tfrac{1}{2}|\langle\kappa; 1|\xi; \kappa\rangle|^2(1 + \cos 4\sqrt{\kappa}gt) + \sum_{M=2}^{\infty} \frac{|\langle\kappa; M|\xi; \kappa\rangle|^2}{(a_M + a_{M-1})^2}$$

$$\times \left\{ (a_M - a_{M-1})\left(a_{M-1} - \frac{a_M}{2}\right) + \frac{a_M}{2}(a_{M-1} - a_M)\cos 2\sqrt{2(a_M + a_{M-1})}gt \right.$$

$$\left. -4a_M a_{M-1}\cos\sqrt{2(a_M + a_{M-1})}gt \right\}. \tag{85}$$

It is easy to show furthermore that

$$|\langle\kappa; n|\xi, \kappa\rangle|^2 = (1 - |\xi|^2)^{2\kappa}\frac{\Gamma(n + 2\kappa)}{n!\Gamma(2\kappa)}|\xi|^{2n} \qquad n \geqslant 0. \tag{86}$$

It is interesting to compare these results with those given in [8, 9] for the one-atom su(1,1) JC model, which is the model equation (2) for $\mathcal{N}_0 = 1$. A similar analysis, albeit simpler, than the one performed above for $\mathcal{N}_0 = 2$, leads to an expression

$$\langle\Phi(t)|S_3|\Phi(t)\rangle = -\frac{1}{2}\sum_{M=0}^{\infty}|\langle\kappa; M|\xi; \kappa\rangle|^2 \cos(2\sqrt{a_M}gt) \tag{87}$$

for the time evolution of the atomic inversion. In this case of $\kappa = \tfrac{1}{2}$, which is that of intensity dependent JC model (cf [8, 10]), the behaviour of the atomic inversion equation (87) becomes strictly periodic, and the result of [8] is recovered as follows

$$\langle\Phi(t)|S_3|\Phi(t)\rangle = -\frac{1}{2}(1 - |\xi|^2)\frac{1 - |\xi|^2\cos(2gt)}{1 - 2|\xi|^2\cos(2gt) + |\xi|^4}. \tag{88}$$

The time evolution of the atomic inversions of the one- and two-atom cases ($\kappa = \tfrac{1}{2}$) are compared in figure 1. It is evident that already for $\mathcal{N}_0 = 2$ the property of exact periodicity of the time evolution of the intensity-dependent model is no longer realized while certain traces of quasiperiodicity can still be observed.

We turn next to the investigation of the squeezing properties of the su(1,1) Perelomov coherent states for the case of two atoms ($\mathcal{N}_0 = 2$). It is convenient to introduce the following field operators

$$X_j = \tfrac{1}{2}e^{-\frac{i\pi}{4}(1+(-1)^j)}(ae^{2i\kappa\omega t} - (-1)^j a^\dagger e^{-2i\kappa\omega t}) \qquad j = 1, 2. \tag{89}$$

As can be readily seen the dispersions of these field operators, i.e. $(\Delta X_j)^2 \equiv \langle X_j^2\rangle - \langle X_j\rangle^2$, calculated with respect to a coherent state $|\alpha\rangle$, $a|\alpha\rangle = \alpha|\alpha\rangle$ of the Heisenberg–Weyl algebra, are $(\Delta X_j)^2 = \tfrac{1}{4}$, $j = 1, 2$. A quantum state exhibits squeezing [26] when the corresponding dispersions of the field operators fulfil the condition $(\Delta X_j)^2 \leqslant \tfrac{1}{4}$, either for $j = 1$ of for $j = 2$. For later convenience we introduce in the following $S_j(t, \kappa) \equiv 4(\Delta X_j)^2 - 1$, such that the squeezing condition becomes

$$S_j(t, \kappa) \leqslant 0 \qquad \text{for } j = 1 \text{ or } j = 2. \tag{90}$$

Evaluating the function S_j for $\kappa = \tfrac{1}{4}$ (the two-photon transition case), we arrive at the result

$$S_j(t, \tfrac{1}{4}) = (-1)^{j+1}B_1(t)\cos\phi + B_0(t) - 1 \tag{91}$$

where ϕ is given by $\xi = |\xi|e^{i\phi}$, and

$$B_0(t) = 2\langle\Phi(t)|K_0|\Phi(t)\rangle$$
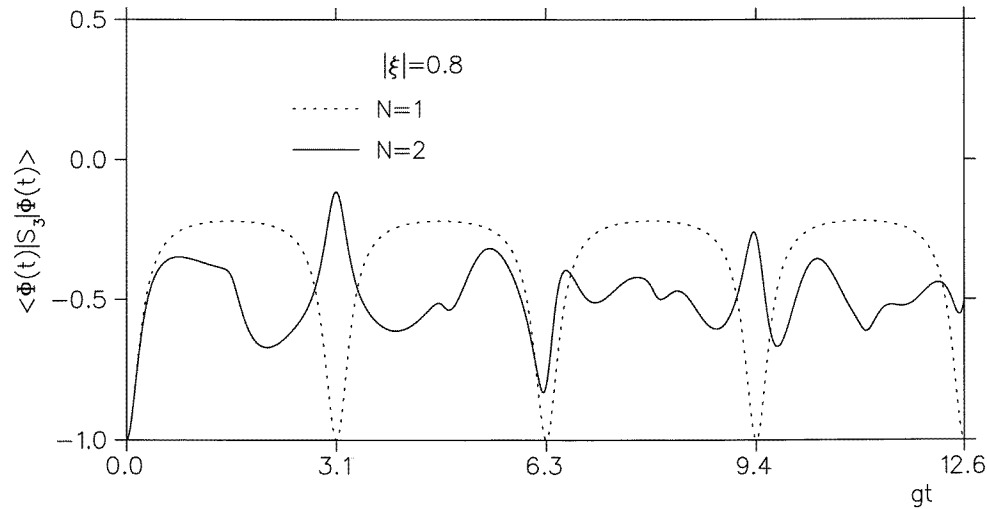$$B_1(t) = 2\langle\Phi(t)|K_\pm|\Phi(t)\rangle e^{\pm i(\phi - \omega t)}. \tag{92}$$

Figure 1. The time evolution of the atomic inversion $\langle \Phi(t)|S_3|\Phi(t) \rangle$ for $\kappa = 0.5$ and $\mathcal{N}_0 = 1$ (dotted curve), and $\mathcal{N}_0 = 2$ (full curve); $g = 1.0$.

Taking into account that $K_0 + S_3$ is an integral of motion, it can be easily shown that

$$B_0(t) = \frac{1 + |\xi|^2}{1 - |\xi|^2} - 2 - 2\langle \Phi(t)|S_3|\Phi(t) \rangle \tag{93}$$

where the last term on the right-hand side is given by equation (85) with $\kappa = \frac{1}{4}$, and

$$B_1(t) = 2e^{-i\omega t + i\phi} \sum_{\substack{M,r \\ j,j'}} \bar{A}_{M+1+r}^{M+1,r,j'} A_{M+r}^{M,r,j} e^{i(E^{M+1,r,j'} - E^{M,r,j})t}$$

$$\times \langle M|\xi \rangle \langle \xi|M+1 \rangle \sum_{n=n_0}^{M} \sqrt{a_{n+1}} \bar{A}_{n+1+r}^{M+1,r,j'} A_{n+r}^{M,r,j}. \tag{94}$$

Note that here $a_n = n(n - \frac{1}{2})$.

After a lengthy algebra making use of the equations (79), (81) and (82), we arrive at the result

$$B_1(t) = \frac{|\xi|}{1 - |\xi|^2} + |\xi|\sqrt{1 - |\xi|^2} \cos(gt)$$

$$+ \operatorname{tr} PQ + \sum_{M \geqslant 2} \operatorname{tr} S(M)T(M) \tag{95}$$

where the matices P, Q, $S(M)$ and $T(M)$ are given by

$$P = \begin{pmatrix} e^{-i(\sqrt{7}+1)gt} & e^{i(-\sqrt{7}+1)gt} \\ e^{i(\sqrt{7}-1)gt} & e^{i(\sqrt{7}+1)gt} \\ e^{-igt} & e^{igt} \end{pmatrix} \tag{96}$$

$$Q = \frac{\sqrt{3}}{2}|\xi|^3\sqrt{1 - |\xi|^2} \begin{pmatrix} \frac{3\sqrt{3}}{14} + \frac{1}{4}\sqrt{\frac{3}{7}} & \frac{3\sqrt{3}}{14} - \frac{1}{4}\sqrt{\frac{3}{7}} & \frac{\sqrt{3}}{14} \\ \frac{3\sqrt{3}}{14} - \frac{1}{4}\sqrt{\frac{3}{7}} & \frac{3\sqrt{3}}{14} + \frac{1}{4}\sqrt{\frac{3}{7}} & \frac{\sqrt{3}}{14} \end{pmatrix} \tag{97}$$

$$S(M)_{j',j} = e^{it(E^{M+1,1,j'} - E^{M,1,j} - \omega)} \qquad j, j' = 1, 2, 3. \tag{98}$$

Here eigenenergies $E^{M,r,j}$ are given by equation (84) with $\kappa = \frac{1}{4}$, while

$$T(M)_{j,j'} = 2\sqrt{1 - |\xi|^2}\frac{\Gamma(M + \frac{1}{2})}{M!\Gamma(\frac{1}{2})}\sqrt{\frac{M + \frac{1}{2}}{M + 1}}|\xi|^{2M+1}t_{j,j'} \qquad j, j' = 1, 2, 3 \tag{99}$$

where elements of the matrix $t_{j,j'}$ are, for $k, k' = 1, 2$,

$$t_{k,k'} = \frac{a_M\sqrt{a_{M+1}}}{4(a_{M+1} + a_M)(a_M + a_{M-1})}(a_{M-1} + (-1)^{k+k'}\sqrt{(a_{M+1} + a_M)(a_M + a_{M-1})} + a_{M+1}) \tag{100}$$

and otherwise

$$t_{k,3} = \frac{a_M\sqrt{a_{M+1}}}{2(a_{M+1} + a_M)(a_M + a_{M-1})}(a_M - a_{M-1})$$

$$t_{3,k'} = \frac{a_{M-1}\sqrt{a_{M+1}}}{2(a_{M+1} + a_M)(a_M + a_{M-1})}(a_{M+1} - a_M) \tag{101}$$

$$t_{3,3} = \frac{2a_Ma_{M-1}\sqrt{a_{M+1}}}{(a_{M+1} + a_M)(a_M + a_{M-1})}.$$

The behaviour as a function of time of the function $S_1(t, \frac{1}{4})$, is given in figure 2. From this figure it is evident that for this case of two atoms quasiperiodic revivals of squeezing can still be observed.

4. Conclusions and discussion

We have formulated and solved exactly, by two methods, the su(1,1) generalization of the TC model. As an example of how the results obtained in this way can be applied to situations with direct physical relevance, we analysed the quantum dynamics and squeezing properties of the Perelomov coherent states in two-photon transitions, as well as the behaviour of atomic inversion in the case of intensity-dependent coupling. As we demonstrated, the
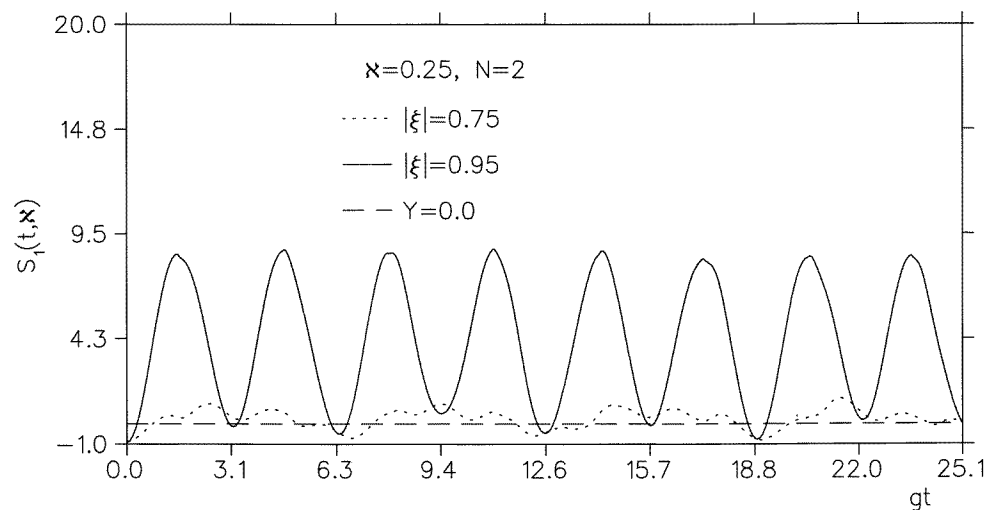

Figure 2. The behaviour as a function of time of $S_1(\frac{1}{4}, t)$ for $\phi = \pi$.

considered quantum dynamics exhibits periodic decays and revivals of squeezing in the case of two atoms. Previously this effect was known only for the one-atom case [8], while our results allow us in principle to analyse the case for any number of atoms $\mathcal{N}_0$ present in the cavity. As reported in [8], which also seems plausible, the greater the squeezing parameter $|\xi|$ the better is the squeezing. However, we have some numerical evidence (cf figure 2) that even for two atoms, under a certain choice of parameters, the field 'lives' longer in the squeezed state for $|\xi| = 0.75$ than it does for $|\xi| = 0.95$. A possible interplay between the number of atoms and squeezing certainly deserves a more detailed analysis and will be reported in a forthcoming publication. This work will provide a good basis for all such work in the future.

One of the methods we applied in this work was the QIM, while the other is the conventional quantum-mechanical approach. Previously the most typical system for which the QIM has been applied, has been models related to two-dimensional statistical mechanics and one-dimensional quantum spin chains [19]. In this work we have extended the very recent attempts [28–30, 12], mostly by ourselves, to apply the QIM in quantum optics. In this approach the problem of construction of the evolution operator that governs the time evolution of the system, is reduced to the solution of a set of the so-called Bethe equations. The solutions of these equations ('rapidities') parametrize the creation operators of the quasiparticles of the system, i.e. collective excitations of the atoms-field system. In general the Bethe equations are transcendental, while in the case considered in this paper they are purely algebraic. To better exemplify our approach we solved the Bethe equations numerically for the case of three quasiparticles. It is worth mentioning that the QIM is most useful in the case of spatially extended system, for which the problem becomes that of quantum field theory. In such situations the QIM would be the only unperturbative approach for an exact evaluation of the quantum dynamics. This is the case, for example, for the quantum Maxwell–Bloch system [28–30]. The su(1,1) TC model is essentially a zero-dimensional model since its physical derivation is based on the assumption that all atoms are located within the characteristic wave length. As indicated above, this fact allows us to solve the model also by the conventional quantum-mechanical approach, which makes it possible to compare the results with those of the QIM. This means that the su(1,1) TC model is a very useful theoretical laboratory to test the methods used. The solution of the Bethe equations is often a difficult numerical problem. The experience obtained from this comparison should help us in the cases [29, 30] when alternative methods do not exist.

It is obvious that the new methods developed here and in [28–30, 12] can open new possibilities for further research by allowing, for example, to evaluate the quantum dynamics of any number of atoms $\mathcal{N}_0$ present in the cavity, and for a variety of possible initial states such as the Barut–Girardello su(1,1) coherent states and their superpositions which also exhibit squeezing [31]. Our results may also be important in the theory of the two-photon micromaser when a high-Q cavity is pumped by a stream of Rydberg atoms such that more than one atom is present in the cavity at any given instant. It is also possible to extend the results of the present work to other cases, which include that of the two-mode bosonic realization of the su(1,1) algebra for which

$$K_+ = a^\dagger b^\dagger \qquad K_- = ab \qquad K_0 = \tfrac{1}{2}(a^\dagger a + b^\dagger b + 1) \qquad \kappa = \tfrac{1}{2}(n + 1). \qquad (102)$$

Here n is the fixed difference in the number of quanta between the two modes. We shall report the results of this and some other extensions of the present work in a forthcoming publication.

Acknowledgments

Appendix

In this appendix we provide the essentials of the algebraic Bethe ansatz method which are necessary for the derivation of the results given in section 2.

The 16 commutation relations between the entries of the matrix equation (21) can be extracted from the bilinear intertwining relation equation (18). These commutation relations include, for example

$$A(\mu)B(\lambda) = f(\mu, \lambda)B(\lambda)A(\mu) + g(\lambda, \mu)B(\mu)A(\lambda)$$
$$D(\mu)B(\lambda) = f(\lambda, \mu)B(\lambda)D(\mu) + g(\mu, \lambda)B(\mu)D(\lambda) \tag{A.1}$$
$$[B(\lambda), B(\mu)] = [D(\lambda), D(\mu)] = [A(\lambda), A(\mu)] = 0$$

for $\mu, \lambda \in \mathcal{C}$. Applying now to the state equation (24),

$$|M, \alpha; \{\lambda_i^\alpha\}_{i=1}^M\rangle = \phi_0^{M,\alpha} \prod_{i=1}^M B(\lambda_i^\alpha)|0\rangle_r \tag{A.2}$$

the operators $A(\mu)$ and $D(\mu)$, and employing the commutation relations (A.1) we arrive at the results

$$A(\mu) \prod_{j=1}^M B(\lambda_j^\alpha)|0\rangle_r = \Lambda \prod_{j=1}^M B(\lambda_j^\alpha)|0\rangle_r + \sum_{n=1}^M \Lambda_n B(\mu) \prod_{\substack{j=1 \\ j \neq n}}^M B(\lambda_j^\alpha)|0\rangle_r \tag{A.3}$$

$$D(\mu) \prod_{j=1}^M B(\lambda_j^\alpha)|0\rangle_r = \tilde{\Lambda} \prod_{j=1}^M B(\lambda_j^\alpha)|0\rangle_r + \sum_{n=1}^M \tilde{\Lambda}_n B(\mu) \prod_{\substack{j=1 \\ j \neq n}}^M B(\lambda_j^\alpha)|0\rangle_r \tag{A.4}$$

where

$$\Lambda = a(\mu) \prod_{j=1}^M f(\mu, \lambda_j^\alpha)$$

$$\Lambda_n = a(\lambda_n^\alpha)g(\lambda_n^\alpha, \mu) \prod_{\substack{j=1 \\ j \neq n}}^M f(\lambda_n^\alpha, \lambda_j^\alpha) \tag{A.5}$$

$$\tilde{\Lambda} = d(\mu) \prod_{j=1}^M f(\lambda_j^\alpha, \mu)$$

$$\tilde{\Lambda}_n = d(\lambda_n^\alpha)g(\mu, \lambda_n^\alpha) \prod_{\substack{j=1 \\ j \neq n}}^M f(\lambda_j^\alpha, \lambda_n^\alpha). \tag{A.6}$$

As can easily be deduced from (A.3), (A.4) with (A.5), (A.6), the state (A.2) is an eigenstate of the trace

$$\tau(\mu) = A(\mu) + D(\mu) \tag{A.7}$$

provided that there is mutual cancellation of the so-called 'unwanted terms', which are the second terms on the right-hand sides of (A.3) and (A.4). This means that the relation

$$\Lambda_n + \tilde{\Lambda}_n = 0 \qquad n = 1, 2, \ldots, M \tag{A.8}$$

should be satisfied.

Using now definitions (A.5) and (A.6) with equation (20), it can be shown that the system of equations (A.8) is nothing but the system of conventional Bethe equations as given by equation (27).

The eigenvalues $\theta^{M,\alpha}(\mu)$ of the eigenstate (A.2),

$$\tau(\mu)|M, \alpha; \{\lambda_i^\alpha\}_{i=1}^M\rangle = \theta^{M,\alpha}(\mu)|M, \alpha; \{\lambda_i^\alpha\}_{i=1}^M\rangle \tag{A.9}$$

are then given by the equation

$$\theta^{M,\alpha}(\mu) = \Lambda + \tilde{\Lambda} \tag{A.10}$$

which is nothing but equation (26) (compare again (A.5), (A.6), and equation (20)).

We can further elaborate on the expression for the eigenfunctions (A.3) in a way similar to what is used in the case of the so-called two-site model (see [19]). Let us split the L-operator equation (21) into a product of the different L-operators $L_1(\lambda)$ and $L_2(\lambda)$, each of which satisfies the same algebra equation (18):

$$\begin{aligned}
L(\lambda) &= L_1(\lambda)L_2(\lambda) \\
&= \begin{pmatrix} A_1(\lambda) & B_1(\lambda) \\ C_1(\lambda) & D_1(\lambda) \end{pmatrix} \begin{pmatrix} A_2(\lambda) & B_2(\lambda) \\ C_2(\lambda) & D_2(\lambda) \end{pmatrix} \\
&= \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}
\end{aligned} \tag{A.11}$$

and possesses its own vacuum vector $|0\rangle_1$ and $|0\rangle_2$, respectively, such that $C_1(\lambda)|0\rangle_1 = 0$ and $C_2(\lambda)|0\rangle_2 = 0$. It is evident that

$$B(\lambda) = A_1(\lambda)B_2(\lambda) + B_1(\lambda)D_2(\lambda). \tag{A.12}$$

Notice that the entries of the L-operators $L_1(\lambda)$ and $L_2(\lambda)$ mutually commute. Substituting the representation (A.12) into (A.3) and applying commutation relations (A.1), we arrive at the result [19]

$$|M, \alpha; \{\lambda_i\}_{i=1}^M\rangle = \phi_0^{M,\alpha} \sum_{\{\lambda_i^\alpha\}_{i=1}^M = \{\lambda_{i_s}^\alpha\}_{s=1}^n \cup \{\lambda_{i_p}^\alpha\}_{p=n+1}^M} \prod_{s=1}^n \prod_{p=n+1}^M a_1(\lambda_{i_s}^\alpha) d_2(\lambda_{i_p}^\alpha) f(\lambda_{i_s}^\alpha, \lambda_{i_p}^\alpha)$$
$$\times B_1(\lambda_{i_p}^\alpha))|0\rangle_1 B_2(\lambda_{i_p}^\alpha)|0\rangle_2 \tag{A.13}$$

where summation over $\{\lambda_i^\alpha\}_{i=1}^M = \{\lambda_{i_s}^\alpha\}_{s=1}^n \cup \{\lambda_{i_p}^\alpha\}_{p=n+1}^M$ is that over all possible decompositions of the set $\{\lambda_i^\alpha\}_{i=1}^M$ into two nonintersecting subsets of n and $M - n$ elements, respectively, for $0 \leqslant n \leqslant M$. Here $a_1(\lambda)$ and $d_2(\lambda)$ are defined such that

$$A_1(\lambda)|0\rangle_1 = a_1(\lambda)|0\rangle_1 \qquad D_2(\lambda)|0\rangle_2 = d_2(\lambda)|0\rangle_2. \tag{A.14}$$

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